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Transition to a wavy vortex régime in convective flow between inclined plates

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Experimental results concerning the supercritical behaviour of instabilities occurring in the thermal convective flow between inclined plates are presented. For inclination δ between 90° and 10° ($\delta = 0°$ is the vertical orientation), with the bottom plate hotter, it is known that the primary instabilities are longitudinal convective rolls with axes oriented up the slope. These rolls are found to become wavy at a supercritical Rayleigh number $Ra(\delta, Pr)$, with an upslope wavelength which seems to be related directly to the wavelength of the original rolls. Measurements indicate that these transitions take place at Reynolds numbers which are probably too small for the process to be attributed to a shear instability. It is thought that the cross-slope derivative of the buoyancy force, which exists because of the tilted geometry, is important in generating the required vorticity. The transition is crucial to the development of violent unsteadiness. The breaking of the waves leads to turbulence at much lower Rayleigh numbers than those required in convection between horizontal plates. The transition to wavy vortices appears to be very similar to that which occurs for Taylor vortices in cylindrical Couette flow.

1. Introduction

In another paper (Hart 1971, hereafter I) an experiment is described in which thermal convection is caused to occur in a differentially heated, inclined box. The geometry and notation for this problem are shown in figure 1. The flows for $D/L \ll 1$ are governed by the following parameters: the Rayleigh number, $Ra = (g\gamma\Delta TD^3)/\kappa\nu$; the Prandtl number, $Pr = \nu/\kappa$; the aspect ratio, $h = D/H$; and the tilt angle δ; which have been defined in the usual manner. In this work $Pr = 6{\cdot}7$ (water) and $h = 0{\cdot}027$. As was discussed in I, this particular problem possesses a rather large selection of primary instabilities which develop on the mean upslope–downslope unicellular circulation. If δ is made slightly less than 90°, the first instabilities to appear as the Rayleigh number is raised are longitudinal convective rolls with axes oriented upslope (x). This same kind of secondary motion is always the first to appear provided $\delta \gtrsim 10°$. The region $10° \lesssim \delta \leqslant 90°$ is convectively unstable, that is to say the correlations between the cross-stream velocity w and the temperature perturbation T are large and positive.

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By observing the motions in plan view it was easy to see that the development of turbulence for $20^\circ < \delta < 90^\circ$ was rather different from what it was in the horizontal, parallel-plate convection limit. In the latter case the initial roll instabilities aligned with the side-walls. As Ra was increased slowly they became more irregular in orientation and developed a striation structure (three-dimensional) near $Ra = 20\,000$. Unsteadiness became evident as Ra approached 50 000. These observations are consistent with the more detailed measurements of Krishnamurti (1970). In the inclined case the transition to turbulence was rather different. The initial longitudinal rolls became wavy at a specific supercritical Ra. As Ra was increased further the waviness assumed larger amplitudes which ultimately led to wave-breaking and turbulence. For $\delta = 45^\circ$ unsteadiness sets in at a Rayleigh number less than 0·2 times that required for similar development at $\delta = 90^\circ$. Figure 2 (plate 1) shows a visualization of the wavy vortices obtained by looking down through a glass lid at a suspension of fish flakes in water.

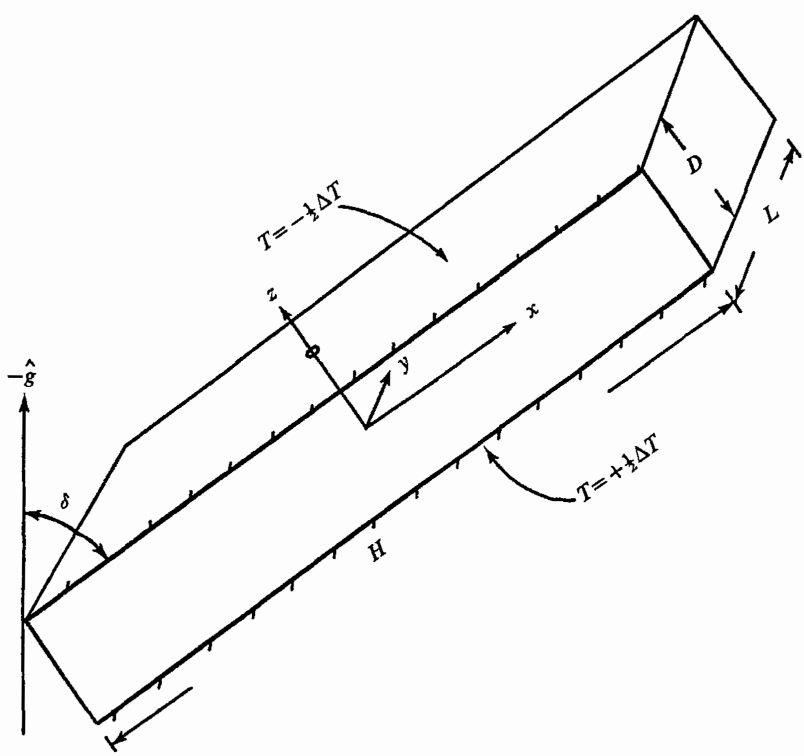

FIGURE 1. Geometry and co-ordinates for the tilted convection experiment.

The onset of wavy vortices has often been observed to precede turbulence in a number of other systems. The most striking example is that occurring in cylindrical Couette flow (Coles 1965). Longitudinal vortices also appear on the single, inclined, heated plate (Sparrow & Husar 1969). Development in this case takes place up the plate (the Rayleigh number is a function of the upslope co-ordinate). In their photographs one can see that the first change in the vortices is that they become wavy. In this example too, the waves break and turbulence follows. Willis & Deardorff (1970) discuss vertically coherent wavy motions observed in horizontal plate convection. In the light of these other studies it is the purpose of this paper to present the transition data for our experiment in the hope that these kinds of motions may be better understood.

2. Experimental observations

In I a technique was described whereby the primary instabilities were detected by looking for waves induced along a dye line at $x = z = 0$. The dye line was injected using the standard thymol blue method of Baker (1966). In order to detect the onset of waviness a similar method was used except now the dye lines were induced along $y = z = 0$; that is, parallel to the axes of the primary longitudinal instabilities. Thus as long as the vortices remain straight, dye will be advected uniformly along the length of the wire. As soon as waviness sets in, non-uniform advection of the dye will cause it to become wavy also. Both the transition points and the wavelengths can be measured in this manner. This method was automated by changing one of the plate temperatures slowly ($< 0{\cdot}1\,^{\circ}\mathrm{C}/20\,\mathrm{min}$) using a clock drive on one of the temperature controllers. For fixed δ the fluid was caused to pass through longitudinal roll onset and on to the wavy régime. Pictures of the dye wire were taken every minute, and a set of lights which contained a binary-code representation of the temperature difference were photographed split-image along with events in the tank. Thus by looking through the film record the wavy transition could easily be identified with a particular Rayleigh number. In identifying the onset of waviness it was required that the secondary motion caused by it persist, as it was sometimes observed that the longitudinal vortices shifted slightly in a transient manner as they adjusted themselves into perfect alignment over the whole length of the tank. For each angle several runs were made and the error bars in the critical Rayleigh number data are based on the scatter, with the points representing the average.

Figure 3 (plate 2) shows a typical sequence of photographs. One can easily see the onset (*b*), finite amplitude effects (*c*), and subsequent breaking and unsteadiness (*d*) due to the meandering. Figure 4 contains the transition data from all of the runs with $\delta > 0$. The solid line denotes the experimentally determined primary transition from the mean unicell to the longitudinal roll régime. The wavy vortex transition has a minimum near $\delta = 45^{\circ}$ and the cut-offs near $\delta = 90^{\circ}$ and $\delta = 10^{\circ}$. The observed wave-numbers are shown in figure 5. The solid line is for the wave-numbers of the longitudinal rolls, multiplied by 0·57. It is seen that the meander wave-numbers k_m are nearly constant multiples of the longitudinal roll wave-numbers, k_r, e.g.

$$k_m \doteq 0{\cdot}57 k_r. \qquad (1)$$

It was observed that the longitudinal rolls which occur for $\delta < 0^{\circ}$ (see I) also develop a wavy structure. The transition points for this were not measured but overhead streak photographs indicate that even for these rolls where k_r is significantly different from 3·1, relation (1) holds, approximately. The dye-line pictures further suggest that the initial wavy pattern is stationary, at least in so far as the sinusoidal dye patterns did not tend to move along the x axis within observation times of a few minutes.

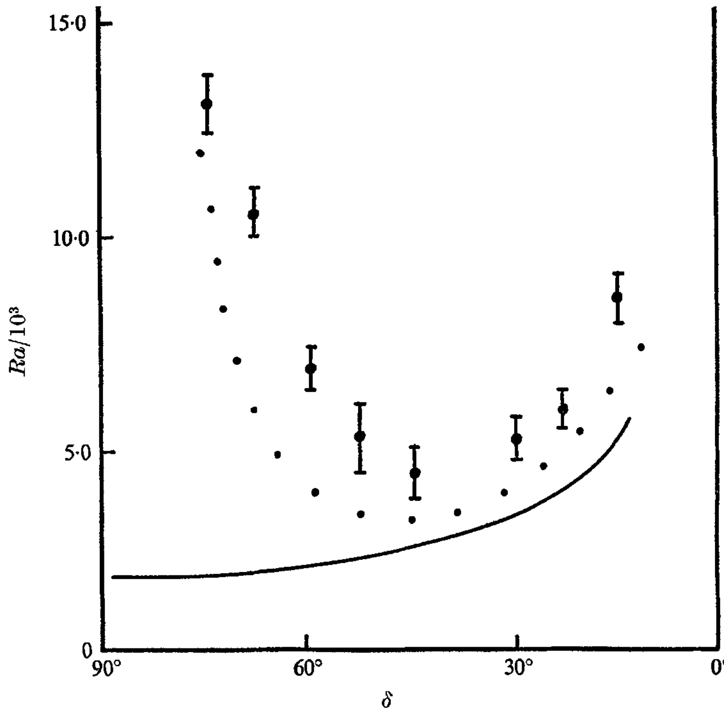

FIGURE 4. Critical Rayleigh numbers (points) for the transition to wavy vortices. The solid line is the experimental critical curve for transition to parallel longitudinal rolls. The dotted curve represents equation (8). In all cases $Pr = 6{\cdot}7$, $h = 0{\cdot}027$.

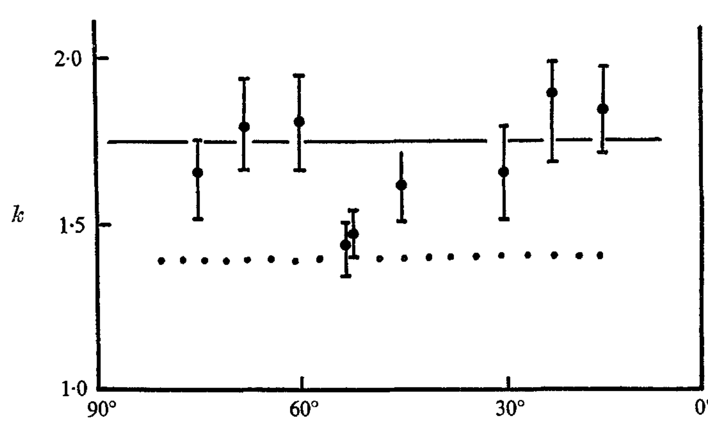

FIGURE 5. Critical wave-numbers for the meanders. The solid line gives the wave-numbers ($\times\, 0{\cdot}57$) for the longitudinal rolls. The dotted curve represents equation (9).

3. Discussion

The form of the meander transition curve, namely the pronounced minimum near $\delta = 45°$, suggests a non-linear interaction involving both local components of buoyancy force, $g\rho \sin\delta\hat{z}$ and $g\rho\cos\delta\hat{x}$. To answer the question of the meander generation one would need to proceed formally and tackle the three-dimensional non-linear problem. Using a generalization of the method of Stuart (1960) and Watson (1960), Davey, DiPrima & Stuart (1968) were able to deduce some properties of the instability of Taylor vortices. In the present problem, however, it would be difficult to carry out a similar analysis since the transition Rayleigh numbers are at least twice the critical Rayleigh number for the rolls, and the usual third-order expansions may not be sufficient. Rather than get involved in complicated mathematics, which would be quite beyond the scope of this paper, we look at two ways in which the combined unicell-roll flow might become unstable. To try and get a rough idea of what is going on we estimate the order of magnitudes involved in the generation of steady z vorticity, periodic in x, and independent of z.

The mean fields which are thought to be important are the y-dependent temperature and velocity fields associated with the equilibrium rolls. We have used the energy method of Stuart (1958) to calculate an equilibrium amplitude for the mean fields. For our purposes this should be reasonably accurate, at least near $\delta = 45°$, where the Rayleigh numbers are not too large. The temperature field is

$$\overline{T} = -z + T_r,$$

where $$T_r \doteq 12(Ra\sin\delta - 1708)^{\frac{1}{2}}\cos\pi z\cos k_r y/Ra\sin\delta. \tag{2}$$

The velocity field is driven by the above temperature field (e.g. $\nabla^2 u + \cos\delta\,.\,T = 0$) and is

$$\overline{U} = \tfrac{1}{6}\cos\delta[z^3 - \tfrac{1}{4}z] + u_r,$$

where $$u_r \doteq 0{\cdot}6\cot\delta(Ra\sin\delta - 1708)^{\frac{1}{2}}\cos\pi z\cos k_r y/Ra. \tag{3}$$

This is valid for $\delta \lesssim 60°$, where upslope buoyancy is much greater than advection.

We now look for ways perturbation meander vorticity $\omega_m = \nabla_H^2\eta^*$ with $u_m = -\eta_y^*$, $v_m = \eta_x^*$, might be generated in a manner consistent with the observations. One possibility is through shear forces creating stationary disturbances on the velocity field (3). These are governed by the Orr–Sommerfeld equation, which for the z averaged u_r, and period fluctuations $\eta^* \sim \eta(y)\,e^{ik_m x}$, is

$$\left(\frac{d^2}{dy^2} - k_m^2\right)^2\eta - ik_m Re\left[\cos k_r y\left(\frac{d^2\eta}{dy^2} - (k_m^2 - k_r^2)\,\eta\right)\right] = 0. \tag{4}$$

The lateral Reynolds number for water is

$$Re \equiv \frac{g\gamma\Delta T D^2}{\nu}\frac{D}{\nu}\,|u_r|_{\max} \doteq 0{\cdot}08\cot\delta(Ra\sin\delta - 1708)^{\frac{1}{2}}. \tag{5}$$

Since the stability equation (4) has periodic coefficients one looks for solutions of the form

$$\eta = e^{\mu y}\phi(y), \quad \phi \text{ periodic in } 2\pi/k_r,$$

with $\mu_r = 0$. We have not tried numerical solutions of this complicated problem but one can obtain a rough estimate of the behaviour by looking at the magnitudes of the source and dissipation terms. We substitute the observed values of $k_m = 1\cdot 7$, $k_r = 3$, and estimate derivatives using $\eta' \sim k_r \eta$. Then

$$Re \sim \frac{(k_m^2 + k_r^2)^2}{k_m^3} \sim O(25).$$

The values of Re for which the meanders are observed are typically $O(1)$. Shear forces may be too weak to initiate instability.

There is, alternatively, the generation of z vorticity by the curl of the buoyancy force, proportional to T_y. This exists here only because of the tilted geometry. It is possible that the buoyancy coupling may provide a means by which the available potential energy stored in the y dependent part of the mean temperature field may be released. One can imagine a situation where vorticity is generated by the buoyancy forces and in which the velocity field associated with this vorticity regenerates the required temperature excess by advection of the mean temperature field $v_m T_{ry}$. In fact a scale analysis on this process alone, e.g. on

$$\nabla_H^4 \eta^* - \cos\delta T_y^* = 0, \tag{6}$$

and

$$\nabla_H^2 T^* + Ra\, \eta_x^* T_{ry} = 0 \tag{7}$$

yields, using (2), a critical curve

$$Ra = 580 \sin\delta/\cos^2\delta + 1708/\sin\delta \tag{8}$$

at a meander wave-number

$$k_m = k_r/\sqrt{5}, \tag{9}$$

which minimizes $Ra(k_m)$. These numbers are in fair agreement with the data, but should only be regarded as suggestive, as it is not at all clear that the terms left out of (6) and (7) are negligible everywhere, or that such a crude scale analysis (in which essentially we have again set $\eta^* \sim e^{ik_r y} e^{ik_m x}$) gives a reasonable approximation to the actual solution of the full problem. However, it does suggest a possible energy transformation mechanism.

Unlike the wavy vortices observed by Willis & Deardorff, the meandering motions here are not vertically coherent, there is a strong cross-stream (z) phase shift. Though the above models are deficient in many respects, perhaps the most important assumption to relax is that of two-dimensional disturbances. An analysis resembling the present one for the shear instability mechanism has been suggested by Davey *et al.* (1968) in an effort to give a physical explanation for the azimuthal wave-number 1 Taylor vortex instabilities. There, also, the primary deficiency of such a theory is the assumed r independence (here z independence) of the meandering. While their results suggested that the periodic axial shear in the mean azimuthal velocity was a possible source for wavy motions, in our case the corresponding cross-slope shear of the upslope velocity appears to be too small, and it is likely that the lateral differential buoyancy force (equivalent to the differential Coriolis force $2\Omega u_z$ in the rotating problem) is a necessary ingredient for the slant convective meanders. Of course our meanders perhaps correspond more closely to the shorter wavelength (azimuthal wave-number 4) meanders observed by Coles (1965).

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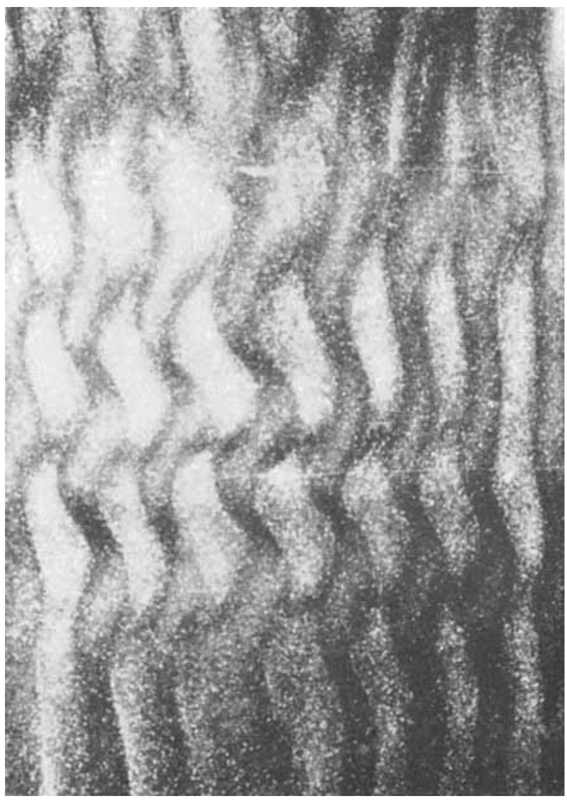
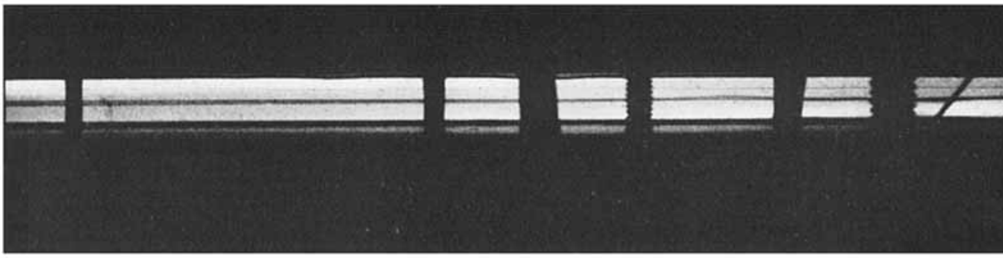
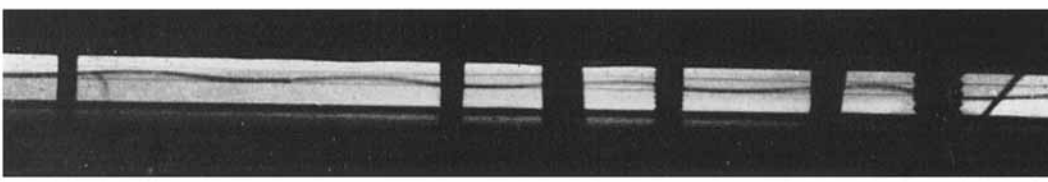


FIGURE 2. Plan view of the wavy vortices. $Ra = 7650$, $\delta = 60°$.



(*a*)



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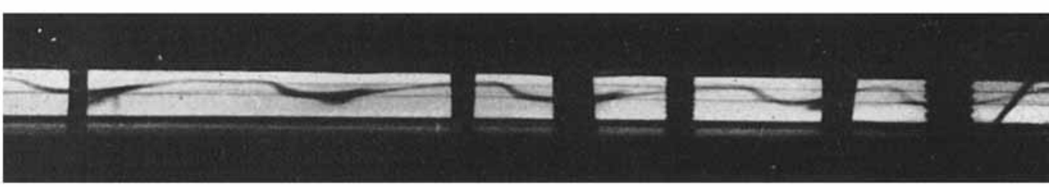
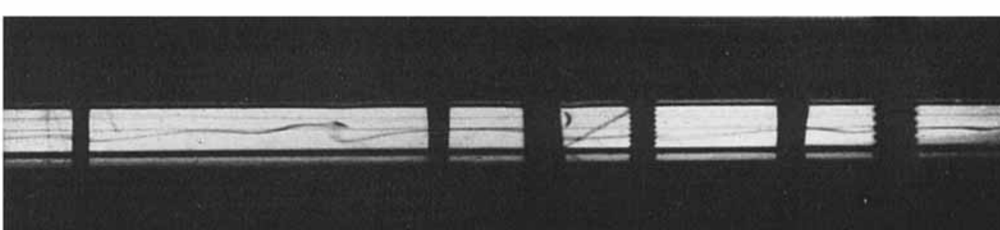


(*c*)



(*d*)

FIGURE 3. Dye-line visualization of the onset of meandering. $Pr = 6{\cdot}7$, $h = 0{\cdot}027$, $\delta = 30°$. In (*a*) $Ra = 4510$, in (*b*), 5160, in (*c*), 5780, and in (*d*), 7570.